

# Baryons and Nuclei in the Large $N_c$ Limit

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## Abstract

The relation between the Skyrme model and the constituent quark model, which appears in the large  $N_c$  limit is described. Examples of similarity in the predicted phenomenology for baryons are shown. Finally the application to nuclei is discussed.

## 1 Introduction

Judah M. Eisenberg was elegant in appearance as well as in expression. His taste for elegance carried over into his research, and hence it was most natural that his attention should also be drawn to Skyrme's topological soliton model for the baryons. He published 21 papers on the application of this model, the mathematical beauty of which transcends its phenomenological utility.

In the non-perturbative region, which comprises most low energy baryon structure and all of nuclear structure, the large color limit of QCD has proven of great utility [1, 2]. In this limit those Feynman diagrams, which have the largest  $N_c$  factor in dominate the  $S$ -matrix. Those diagrams involve only planar gluons, which then may be replaced by  $q\bar{q}$  pair lines, and as a consequence all surviving diagrams admit a meson exchange interpretation.

In the large  $N_c$  limit counting of the  $N_c$ -factors reveals mesons to be stable and non-interacting. Baryons are different. They are color singlets of  $N_c$  quarks, and accordingly their mass scales as  $N_c\Lambda_{QCD}$ , where  $\Lambda_{QCD}$  is the (only) dimensional QCD scale factor. Their radius remains proportional to  $1/\Lambda_{QCD}$  and  $N_c$  independent. Accordingly the quark density with the baryon grows beyond bound, and a Hartree approximation becomes appropriate [3]. Finally as the meson-baryon coupling constants in general are proportional to  $\sqrt{N_c}$ ,

mesons are strongly coupled to baryons in the large  $N_c$  limit.

There are two approaches to describe the baryons in a large  $N_c$  limit. The first is to employ a constituent quark model description based on appropriately symmetrized products of  $N_c$  quark wave functions in the Hartree approximation. The other is to construct the baryons as topologically stable soliton solutions to a chiral Lagrangian of meson fields, with the general form [2]

$$\mathcal{L}_{meson} = N_c \mathcal{L}_p\left(\frac{\phi}{\sqrt{N_c}}\right), \quad (1)$$

where  $\mathcal{L}_p$  is polynomial of meson fields and their gradients, which satisfies the (nontrivial) stability requirements. The Skyrme model [4], and its generalizations [5, 6] are generic examples of the latter approach.

## 2 Large $N_c$ Operator Algebra

Among mesons the pions stand out because their role as the Goldstone bosons of the spontaneously broken approximate chiral symmetry of QCD. As such their coupling to hadrons has to vanish with 4-momentum. The pion coupling to fermions is accordingly

$$\mathcal{L} = \frac{1}{f_\pi} A^{\mu a} \partial_\mu \pi^a, \quad (2)$$

where  $f_\pi$  is the pion decay constant ( $\sim \sqrt{N_c}$ ), and  $A^{\mu a}$  is the axial vector of the fermion, which in the case of a baryon scales as  $N_c$ .

In the large  $N_c$  limit, the baryons propagator reduces to the static propagator, and  $\pi$ -baryon scattering becomes recoilless. Consequently the  $\pi$ -baryon scattering amplitude only involves bilinear combinations of the space components of the axial current, from which it proves convenient to separate a factor  $N_c$  as

$$A^{ia} = g N_c X^{ia}, \quad (3)$$

where  $g$  is a coupling constant introduced for convenience.

For the operators  $X^{ia}$  the following  $N_c$  expansion proves useful [7]:

$$X^{ia} = X_0^{ia} + \frac{1}{N_c} X_1^{ia} + \frac{1}{N_c^2} X_2^{ia} + \dots \quad (4)$$

The lowest order operator  $X_0^{ia}$  and the spin- and isospin operators then satisfy a contracted SU(4) algebra. This is the algebra of the Skyrme model, which is made explicit by the identification

$$X_0^{ia} = \frac{1}{2} Tr\{A\sigma^i A^\dagger \tau^a\}, \quad (5)$$

A being the SU(2) rotational collective coordinate used in the spin-isospin quantization of the Skyrme Hamiltonian [8].

Baryon operators may then be described systematically in terms of the operators  $X_n^{ia}$  or alternatively in terms of the SU(4) generators spin, isospin and spin-isospin  $G^{ia}$ , where the relation to the  $X_n^{ia}$ :s is given by the limiting relation

$$\lim_{N_c \rightarrow \infty} \frac{G^{ia}}{N_c} = X_0^{ia}. \quad (6)$$

The connection to the constituent quark model obtains by expression of the SU(4) generators in terms of corresponding quark operators:

$$J^i = \sum_{l=1}^{N_c} q_l^\dagger \frac{\sigma^i}{2} q_l, \quad I^a = \sum_{l=1}^{N_c} q_l^\dagger \frac{\tau^a}{2} q_l, \\ G^{ia} = \sum_{l=1}^{N_c} q_l^\dagger \frac{\sigma^i}{2} \frac{\tau^a}{2} q_l. \quad (7)$$

Thus in the large  $N_c$  limit the constituent quark model and the Skyrme model give equivalent results.

### 3 Phenomenological considerations

The formalism outlined above may be employed to derive systematic  $1/N_c$  expansions for baryon operators as masses, magnetic moments [2]. Relations between the available two-body operators allows expression of the baryon mass operator as

$$M = m_0 N_c + m_2 \frac{1}{N_c^2} \vec{J}^2 + m_3 \frac{1}{N_c^3} \vec{J}^4 + \dots, \quad (8)$$

where  $\vec{J}$  is the spin-operator. To order  $1/N_c^2$  this agrees with the Skyrme model result [8]

$$M = M_s + \frac{1}{2\Omega_s} \vec{J}^2, \quad (9)$$

where  $M_s$  and  $\Omega_s$  are the mass and moment of inertia of the soliton.

Returning to the quark model representation, (7) it has been found that among the two-body operator combinations of  $J^i$ ,  $I^a$  and  $G^{ia}$ , the only significant combination for the mass operator is the operator

$$-\sum_{i<j}\vec{\sigma}_i\cdot\vec{\sigma}_j\vec{\tau}_i\cdot\vec{\tau}_j, \quad (10)$$

and its generalization SU(3) flavor [9, 10]. This is consistent as it is this operator and the scalar unit operator, which have the highest  $N_c$ -scaling factors [11]. As this operator appears in the pion exchange and multipion exchange interaction between constituent quarks, it suggests that in the region between the chiral restoration scale  $\Lambda_\chi^{-1} \sim 1/4f_\pi$  and the confinement scale  $1/\Lambda_{QCD}$  the effective dynamical description of baryons is in terms of constituent quarks that interact by exchanging pions [12].

The reason for the effectiveness of the operator (8) in organizing the baryon spectrum in agreement with the empirical one is that it is the only operator combination that is able to achieve the correct ordering of the positive and negative parity states in the spectrum. Any monotonic confining interaction would order the states in baryon spectrum in shells of alternating parity. Yet the lowest excited states in the nucleon spectrum are successively the  $N(1440)$ ,  $\frac{1}{2}^+$  and the  $N(1535)$ ,  $\frac{1}{2}^-$ . As these states (and the nucleon) all have mixed color-spin symmetry  $[21]_{CS}$ , the color-spin dependent perturbative gluon exchange model for the hyperfine interaction cannot reverse their normal ordering and bring them into agreement with the empirical ordering. On the other hand the  $N(1535)$ ,  $\frac{1}{2}^-$  state has mixed flavor-spin symmetry  $[21]_{FS}$ , whereas the nucleon and the  $N(1440)$ ,  $\frac{1}{2}^+$  state have complete flavor-spin symmetry  $[3]_{FS}$ . Therefore, in combination with a sufficiently strong orbital matrix element, the operator (8) is able to reverse the normal ordering, and bring the spectrum into agreement with experiment [13]. This argument carries over to all flavor sectors of the baryon spectrum, with exception of the  $\Lambda$ -spectrum, where the negative flavor singlet  $\Lambda 1405 - \Lambda 1520$  is the lowest multiplet, which however, also only can be understood with the operator form (8).

The interaction (8) may be interpreted as being due to pion [13] and multipion exchange between the constituent quarks. Two-pion exchange is required to cancel out the tensor interaction of one-pion exchange, as the tensor interaction would otherwise imply substantial spin-orbit splittings in the  $P$ -shell, in

conflict with experiment [14]. With suitable parameter choices the inclusion of two-pion exchange also allows a cancellation of the spin-orbit interaction that arises with a linear scalar confining interaction.

The constituent quark model, with quarks interacting by pion exchange, is usually referred to as the chiral quark model. The dynamical interpretation resembles that of the Skyrme model. Hence, and in view of the discussion in section 2 above, it is no surprise that the reversal of the normal ordering of the states in the baryon spectrum also obtains in the Skyrme model. This model has long been known to imply very low lying vibrational states [15, 16].

## 4 Heavy Flavor Baryons and Pentaquarks

The quark model describes baryons as 3-quark systems. The Skyrme model describes baryons as formed of  $(q\bar{q})^n$  - i.e. mesons - in the field of a soliton, which carries the baryon number. Hyperons are then best described as bound states of soliton and  $K$ ,  $D$  and  $B$  mesons respectively [17, 19].

This provides a unified description of normal hyperons and "pentaquarks". While a hyperon is described as a bound state of a soliton and a heavy flavor meson, the corresponding pentaquark is described as a bound state of soliton and the corresponding heavy flavor antimeson [20]. Note that the nomenclature is due to the non-symmetric description by the constituent quark model of the former as  $qqQ$  states and the latter as  $qqqq\bar{Q}$  states, where  $q$  and  $Q$  denote light and heavy flavor quarks respectively.

Both the chiral quark model, extended to broken  $SU(N_F)$  [18], and the bound state soliton model [19] describe the ground state heavy flavor hyperons well [21]. But only the latter yields absolute predictions for the pentaquark energies without further parameters or assumptions, as the difference between hyperon and pentaquark energies arises solely from the sign of the Wess-Zumino term in the effective meson-soliton interaction [17, 20].

The existence or non-existence of pentaquarks is interesting, because it depends very much on the form of the hyperfine interaction between quarks. If that interaction has the form of perturbative gluon exchange, the lowest energy pentaquark contains a strange quark, and has negative parity [22, 23]. The bound state soliton model [20] and the chiral quark model [24] in contrast predict that the lowest energy - and definitely stable - pentaquark is non-strange and has positive parity. Only the former has been searched for experimentally,

although with somewhat inconclusive results [25]. Experimental identification of the latter would be very informative, and would automatically explain the non-existence of the  $H$ -particle, which hasn't been found in spite of extensive effort [26].

## 5 Nuclei

Most of Judah Eisenberg's work on applications of the Skyrme model dealt with nuclei. This work employed Skyrme's product ansatz for solitons with baryon number larger than 1 [27]:

$$U(r; \vec{r}_1, \dots, \vec{r}_A) = \prod_{i=1}^A U(\vec{r} - \vec{r}_i) \quad (9)$$

When this ansatz is inserted into the Lagrangian density of the Skyrme model, the Lagrangian separates into a sum of single nucleon terms and interaction terms involving 2, 3, ...,  $A$  nucleons. When integrated over  $\vec{r}$  the latter yield models for the 2-, 3-, ...,  $A$ -nucleon interactions. For large internucleon separations these terms have the same form as the corresponding pion-exchange interactions that obtain with more conventional chiral Lagrangians [28].

This is illustrated by the derivation of the three-nucleon interaction based on the product ansatz by Eisenberg and Kälbermann [29]. For large interparticle separations this interaction reduces to the conventional two-pion exchange three-nucleon interaction, that involves an intermediate  $\Delta_{33}$  resonance in the sharp resonance approximation [30]. The appearance of the sharp resonance propagator  $-i(m_\Delta - m_N)$ , in this expression is a direct consequence of the large  $N_c$  mass formula (9).

The product ansatz (9) does not, however, provide a good approximation to the minimum energy configuration of 8 skyrmions with  $B > 1$ . The minimum energy solutions have an interesting topology, which hitherto has not yielded to analytical treatment [31, 32]. This situation has now changed by the discovery of elegant rational map approximations to the minimal energy solutions [33].

The rational map approximation for the  $B = n$  skyrmion takes the form

$$U(\vec{r}) = e^{i\vec{\tau} \cdot \vec{\pi}_n F(r)}, \quad (10)$$

where the "chiral angle"  $F(r)$  depends only on the distance to the center, and  $\vec{\pi}_n$  is defined as

$$\vec{\pi}_n = \frac{1}{1 + |R_n(z)|^2} \{2\text{Re}[R_n(z)], \quad 2\text{Im}[R_n(z)], \quad 1 - |R_n(z)|^2\}. \quad (11)$$

Here  $R_n(z)$  is the rational map for baryon number  $n$ , and  $z$  is defined as  $z = \tan v/2 e^{i\rho}$ . For  $n = 1$ ,  $R(z) = z$ , and (10) reduces to Skyrme's hedgehog solution. For larger  $n$   $R(z)$  are simple rational functions of  $z$ , the simplest case being  $n = 2$  for which  $R_2(z) = z^2$ .

The rational maps open the door to more realistic applications to nuclei based on the Skyrme model, than what hitherto has been possible. As an example the question of the existence of bound states between  $\eta$ -mesons and nuclei may be addressed and shown to be likely [34]. What is still wanting, however, are functional forms that connect the rational maps approximations (10) to the product ansatz (9), which is appropriate for large separations.

## 6 Judah

Judah would have enjoyed the elegance of the rational maps. Of his style in expression, here he is in a letter, dated November 12, 1973, in Charlottesville: "Stanley Hanna and I were very much hoping that you might be able to see your way clear to give a talk at the [1974 photonuclear Gordon] conference. In particular, we wondered if you would be willing to review ... . Of course, it would be appropriate to slant your discussion towards aspects of this problem, which would be of particular relevance for the community of photonuclear physicists". Obviously I accepted. That photonuclear conference, which Judah chaired, will be long remembered by the participants, as one of the (unscheduled) evening talks was the televised resignation speech of R. M. Nixon.

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